ELLIPTIC EQUATIONS WITH VARIABLE NONLINEARITY

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We study the Dirichlet problem for the elliptic equations with variable anisotropic nonlinearity

$$(1) \qquad -\sum_{i=1}^{n} D_i \left(a_i(x,u)|D_i u|^{p_i(x)-2} D_i u\right) - c(x,u)|u|^{\sigma(x)-2}u = f(x),$$

$$(2) \qquad -\sum_{i=1}^{n} D_i \left(a_i(x,u)|u|^{\alpha_i(x)} D_i u\right) - c(x,u)|u|^{\sigma(x)-2}u = f(x) \quad \text{in } \Omega \subset \mathbb{R}^n.$$

The boundary $\partial\Omega$ is Lipschitz–continuous. The coefficients a_i, c and the exponents of nonlinearity $p_i(x)$, $\alpha_i(x)$, $\sigma(x)$ are prescribed functions of their arguments. Equations of the type (1), (2) emerge from the mathematical modelling of various physical phenomena, e.g., processes of image restoration, flows of electro–rheological fluids, thermistor problem, filtration through inhomogeneous media.

Existence of solutions [1,2]. We prove that under suitable restrictions on the coefficients and the nonlinearity exponents the Dirichet problem for equations (1) and (2) admit a.e. bounded weak solutions which belong to the anisotropic analogs of the generalized Lebesgue–Orlicz spaces. The solution of equation (1) is constructed as the limit of a sequence of Galerkin's approximations. We claim that $1 < p_i(x) < \infty$, $1 < \sigma(x) < \infty$, $0 < a_0 \le a_i(x,u) < \infty$, $0 \le c(x,u) < \infty$ and that the exponents $p_i(x)$ are continuous with logarithmic module of continuity.

The existence of a.e. bounded weak solution of equation (2) is proved by means of the Schauder fixed point principle. It is requested that $D_i\alpha_i(x) \in L^2(\Omega)$ and either $\alpha_i(x) \ge 0$ in Ω and $c_0 \ge 0$, or $\alpha_i(x) \in (-1,\infty)$ and $c_0 > 0$.

Uniqueness of solutions [2,3]. It is shown that a.e. bounded (small) solution of equation (1) is unique if $a_i(x,u) \equiv a_i(x)$ and either $p_i(x) \in (1,\infty)$, $\sigma(x) \in (1,2]$ and $c(x,s)$ is Lipschitz–continuous with respect to s, or if $\frac{2n}{n+2} < p_i(x) \le 2$, $\sigma(x) \in [2,\infty)$ and $c(x,u) \ge 0$.

For equation (2), uniqueness of bounded solutions is proved under the assumptions $c(x,u) \equiv 0$, $\alpha_i(x) \equiv \alpha(x) \in (-1,\infty)$, $\nabla\alpha(x) \in L^2(\Omega)$, $\sup_{x\in\Omega, s\in\mathbf{R}} |\nabla_x a(x,s)| < \infty$.

Localization of solutions caused by the diffusion–absorption balance [2]. Localization (or vanishing on a set of nonzero measure) is an intrinsic property of solutions to nonlinear elliptic equations. It is known [4] that for the solutions of equations of the type (1), (2) with constant (but possibly anisotropic) nonlinearity such an effect appears due to a suitable balance between the diffusion and absorption terms of the equation. We show that the same is true for

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equations with variable exponents of nonlinearity. The proof relies on the method of local energy estimates [4, Chapter 1].

Directional localization caused by anisotropic diffusion [1,3]. It is known that for the solutions of nonlinear equations of the type diffusion–absorption the following alternative holds: if u is a nonnegative weak solution of the exterior Dirichlet problem for the equation $-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + u^{\sigma-1} = 0$ with constant exponents p and σ, then

$$\begin{array}{lcl} 1 < p \leq \sigma & \iff & \text{the strong maximum principle holds,} \\ 1 < \sigma < p & \iff & \text{the support of the solution is compact.} \end{array}$$

It turns out to be that this alternative is no longer true if the diffusion operator is anisotropic. In this case the support of the solution may be compact even in the absence of the absorption part. Similar assertions hold for the solutions of equation (1). The property of directional localization allows one to solve problems posed on unbounded domains without additional conditions at infinity. Sufficient conditions of solvability read as restrictions on the asymptotic shape of the domain $\Omega = \{x_1 \in \mathbb{R}_+\} \times \omega(x_1)$ as $x_1 \to \infty$. For example, in the case $n = 2$ it is sufficient to claim that Ω is contained in a cone of aperture less than $\pi/2$.

Most of results extend to solutions of equations of the type (1), (2) with the first–order terms (convection) and to systems of equations of similar structure [2,3].

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